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1 SAMPLE-PATH SOLUTIONS FOR SIMULATION OPTIMIZATION PROBLEMS AND STOCHASTIC VARIATIONAL INEQUALITIES

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Abstract: In this paper, we give an overview of some recent developments in using simulation together with gradient estimation techniques to provide so-

lutions for difficult stochastic optimization problems and stochastic variational inequalities. The basic idea is to observe a fixed sample path (by using the method of common random numbers from the simulation literature), solve the resulting deterministic problem using fast and effective methods from nonlinear programming, and then use the resulting solutions to infer information about the solution of the original stochastic problem. We describe these so-called *sample-path* methods precisely, review some conditions under which they are known to work, and comment on their potential advantages and limitations. We also illustrate some application areas in which these ideas have been successful.

1.1 INTRODUCTION

This expository paper surveys several aspects of *sample-path* methods for solving stochastic optimization and equilibrium problems. Our intent is to describe the methods, show what theoretical support is currently available for convergence, and present a selection of application problems on which the methodology has been successful. We begin in this introduction by giving a brief explanation of the methods and referring to some relevant papers in the literature. In succeeding sections we develop the methods in more technical depth and illustrate some of their applications.

We are interested in solving a problem of optimization or equilibrium, involving a limit function f_∞ which we cannot observe; however, we can observe functions f_n that almost surely converge pointwise to f_∞ as $n \to \infty$. In the kind of applications we have in mind, f_∞ is typically a steady-state performance measure of a dynamic system or an expected value in a static system.

We focus on two types of problems. The first involves optimization; in this case the f_n are extended-real-valued functions:

$$f_n : \mathbf{R}^k \to \mathbf{R} \cup \{\pm\infty\} \text{ for } 1 \leq n \leq \infty,$$

and we are interested in solving

$$\min_x f_\infty(x).$$

This setup also covers constrained optimization problems since we can always set $f_\infty(x) = +\infty$ for x that do not satisfy the constraints.

The second problem type is a variational inequality; in this case the f_n are vector-valued functions:

$$f_n : \mathbf{R}^k \to \mathbf{R}^k \text{ for } 1 \leq n \leq \infty,$$

and our aim is to find a point $x_0 \in C$, if any exists, satisfying

$$\text{for each } x \in C, \quad \langle x - x_0, f_\infty(x_0) \rangle \geq 0, \tag{1.1}$$

where $\langle y, z \rangle$ denotes the inner product of y and z, and C is a polyhedral convex subset of $\mathbf{R}^k$.

Such problems commonly arise in the study of complex stochastic systems. The first is a well known problem of stochastic optimization. Examples of this problem arise from queueing networks, manufacturing systems, communication networks, and financial planning, among other areas. The second type of problem can be used to model various equilibrium phenomena in physics, mathematics, operations research, and economics. If one takes the set C to be the whole space, this problem reduces to solving k equations in k unknowns (*i.e.,* to finding a zero of f_∞). If f_∞ is the gradient of another function then by solving the variational inequality (1.1) we find a point x_0 that satisfies the first order optimality conditions of an optimization problem with C being the feasible set. Other engineering and economic applications of variational inequalities include traffic modeling, Nash equilibrium, and network equilibrium problems; see Ferris and Pang (1997) for a recent survey. In particular, with this setup one can model stochastic equilibrium problems involving expectations or steady-state functions.

In most cases, we have in mind using simulation to observe the f_n's. In systems that evolve over time, we simulate operation of the system for n time units and then compute an appropriate performance measure, whereas in static systems we repeatedly observe instances of the system and compute an average. In both cases, to observe f_n at different parameter settings we use the method of common random numbers. Furthermore, in many cases derivatives or directional derivatives of the f_n can be obtained using well-established methods of gradient estimation such as infinitesimal perturbation analysis; see Suri (1989), Ho and Cao (1991), and Glasserman (1991). The sample-path methods then solve the resulting deterministic problem (using f_n with the fixed sample path selected), and taking the solution as an estimate of the true solution. In Sections 1.2 and 1.3 we explain these ideas more rigorously.

For problems of the first type, the method of stochastic approximation and its variants have been available since the 1950s. Although very prominent, these methods suffer from serious drawbacks such as slow convergence, lack of a stopping criterion, and difficulty in enforcing feasibility (since the method handles constraints via projection onto the feasible set). In addition, the numerical performance of stochastic approximation depends heavily on the choice of a predetermined parameter; see L'Ecuyer *et al.* (1994) for example. The sample-path optimization method proposed in Plambeck *et al.* (1993, 1996) and analyzed in Robinson (1996) overcomes several of these difficulties. In Section 1.2 we describe the basic ideas behind this method, its important features, and its convergence properties, in more detail.

The second type of problem (in its deterministic form), has attracted interest since the 1960s. A survey of developments in the area up to 1990 can be found in Harker and Pang (1990). Josephy (1979) proposed a method of Newton type for such problems, and more recently many variants have been studied. A particularly successful implementation of this idea is the PATH solver of Dirkse and Ferris (1995a).

In the stochastic setting, this problem seems not to have been much studied, with the exceptions of papers by Haurie *et al.* (1987) and De Wolf and Smeers (1997). Both these works are concerned with an equilibrium model of the European gas market and they use discrete scenario representations to model the uncertainty in the system. This approach entails data management problems: see Mulvey and Vladimirou (1991) for example. In Section 1.3, we describe an extension of the sample-path optimization method to solve stochastic variational inequalities. This new form was proposed in Gürkan *et al.* (1996); supporting analysis and an implementation on the European gas market example are in Gürkan *et al.* (1997). In contrast to the scenario approach used in Haurie *et al.* (1987) and De Wolf and Smeers (1997), we used simulation together with gradient estimation to solve this problem.

1.2 SAMPLE-PATH OPTIMIZATION

In this section we describe the sample-path optimization method in the form proposed in Plambeck *et al.* (1993, 1996) and analyzed in Robinson (1996). This method finds an approximate minimizer of a function f_∞ that is an almost-sure limit of a computable sequence of random functions f_n. We present briefly the basic ideas and main convergence result, and comment on some of their implications.

Many problems in simulation optimization can be modeled by an extended-real-valued stochastic process $\{f_n(x) \mid n = 1, 2, \dots\}$. The f_n take values that may be real numbers or $\pm\infty$, whereas the parameter x takes values in $\mathbf{R}^k$. As mentioned earlier, using extended-real-valued random variables is very convenient for modeling constraints, since we can always set $f_n(x) = +\infty$ for those x that do not satisfy the constraints. Furthermore, we generally exclude $-\infty$ by requiring the functions to be *proper*: that is, never $-\infty$ and not everywhere $+\infty$. For each $n \geq 1$ and each $x \in \mathbf{R}^k$, $f_n(x)$ is a random variable defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume the existence of a limit function f_∞ such that the f_n almost surely converge pointwise to f_∞ as $n \to \infty$. For the systems we are concerned with, such existence and convergence can often be inferred from regeneration theorems and/or the strong law of large numbers. In the following we refer to $f_n(x)$ as the sample function and we write $f_n(\omega, x)$ when we want to emphasize the dependence of $f_n(x)$ on the sample point ω.

Let us demonstrate this setup with a simple example. Suppose that we are analyzing an $M/M/1$ queue and we are interested in the steady-state system time of a customer, denoted by f_∞. Let f_n be the average of the system times of n customers, *i.e.*, f_n is the output of a simulation of run length n (n service completions in this case). From regeneration theorems we know that under certain conditions on the parameters of the system f_∞ exists and the f_n converge pointwise to f_∞ along almost every sample path. Therefore in this case the sample-path approach is applicable.

As mentioned earlier, we are interested in finding the infimum and, if it exists, a minimizer of f_∞. In general we can only observe f_n for finite n. Therefore we will approximate minimizers of f_∞ using such information about f_n. The method is simple: we fix $\omega \in \Omega$ and compute a minimizer $x_n^*(\omega)$ of $f_n(\omega, .)$ for large n, then take $x_n^*(\omega)$ as an approximate minimizer of $f_\infty(\omega, .)$. Of course, minimizers of $f_\infty(\omega, \cdot)$ will generally depend on the sample point ω. However, in many practical problems in which one would anticipate using this technique f_∞ is a deterministic function, for example a steady-state performance function or an expected value; *i.e.*, it is independent of ω.

This form of the method was proposed and analyzed in Plambeck *et al.* (1993, 1996) for use with infinitesimal perturbation analysis (IPA) gradient estimates; convergence of the general method is analyzed in Robinson (1996). Similar ideas were proposed by Rubinstein and Shapiro (1993) for use with the likelihood-ratio (LR) method, and these methods are also closely related to the retrospective optimization proposals of Healy and Schruben (1991) and to M-estimation and other techniques. Robinson (1996) gives a brief survey of these and other similar ideas that have appeared in the literature.

There are two key points: (*i*) once we fix n and a sample point ω, $f_n(\omega, x)$ becomes a deterministic function of x; (*ii*) IPA — when it applies — gives *exact* gradients of the f_n. With these observations, very powerful methods of constrained and unconstrained deterministic optimization become available for use on the f_n. In the smooth case we can apply superlinearly convergent methods like the BFGS algorithm (or a variant for constrained problems) to minimize f_n to high accuracy in relatively few function and gradient evaluations. For more information on these algorithms see Fletcher (1987) and Gill *et al.* (1981) and for the software available see Moré and Wright (1993). Using superlinearly convergent methods enables us to be confident about the location and the accuracy of the minimizer of f_n, because we can differentiate between the errors due to the approximation of f_∞ by f_n and those due to the inaccurate computation of a minimizer of f_n. With slower algorithms like stochastic approximation this is difficult if not impossible.

If the sample function and/or performance function we want to minimize is nondifferentiable and convex, then we can use the Bundle-Trust (BT) method.

This is a nonsmooth convex minimization technique of the bundle class, proposed in Zowe (1989) and analyzed in Schramm and Zowe (1990). A closely related method was presented by Kiwiel (1990). For a more extensive extensive discussion of methodology and theory in nondifferentiable optimization, see Hiriart-Urruty and Lemaréchal (1993a, 1993b) and Correa and Lemaréchal (1993). We emphasize that in both the smooth and the nonsmooth case, the deterministic solution methods available can handle constraints explicitly: general smooth nonlinear constraints in the smooth case and linear equations and inequalities in the nonsmooth case.

Another useful feature of this approach is its modularity; the computation of function and gradient values is separated from the optimization. This enables the use of already existing simulation codes (if they also provide gradient values or can be modified to do so), together with optimization codes that call external subroutines for function and gradient evaluations. If the system simulated is large and complex, and the optimization code is sophisticated, then the advantage of modularity becomes more substantial.

To give a precise statement of the convergence properties of the sample-path optimization method, we need to introduce a few concepts. It is well known that ordinary pointwise convergence of a sequence of functions is not sufficient to guarantee that their minimizers converge to a minimizer of the limit function. To ensure the convergence of the minimizers we need to impose a special type of convergence, namely *epiconvergence*, on the sample functions f_n; see Kall (1986) for example.

Definition 1 *A sequence f_n of extended-real-valued functions defined on $\mathbf{R}^k$* epiconverges *to an extended-real-valued function f_∞ defined on $\mathbf{R}^k$ (written $f_n \xrightarrow{e} f_\infty$) if for each $x \in \mathbf{R}^k$ the following hold:*

a. For each sequence $\{x_n\}$ converging to x, $f_\infty(x) \leq \liminf_{n\to\infty} f_n(x_n)$.

b. For some sequence $\{x_n\}$ converging to x, $f_\infty(x) \geq \limsup_{n\to\infty} f_n(x_n)$.

Note that in (b) we actually have $f_\infty(x) = \lim_{n\to\infty} f_n(x_n)$, because of (a).

The sample-path optimization method finds a minimizer of $f_n(\omega, \cdot)$ for fixed ω and fixed large n. If we knew that $f_n \xrightarrow{e} f_\infty$ as $n \to \infty$, then we could assert that for large n, the minimizer of f_n found by the method is close to a minimizer of f_∞. This is a rough statement of the convergence properties of the method. Before we give the precise statement we need to introduce more notation and give some definitions. Among these a crucial concept is that of *complete local minimizing set* introduced in Robinson (1987).

Definition 2 *Let Z be a topological space and let f be an extended-real-valued function on Z. A nonempty subset M of Z is a* complete local minimizing

(CLM) set *for f with respect to an open set $G \supset M$, if the set of minimizers of f on $\operatorname{cl} G$ is M.*

The concept of CLM set extends the idea of isolated local minimizer to cases in which the set of minimizers might not be a singleton.

Let S and T be subsets of $\mathbf{R}^k$. We use the notation $e(S,T)$ for the *excess* of S over T, defined by

$$e(S,T) = \sup_{s \in S} d(s,T); \qquad d(s,T) = \inf_{t \in T} \|s - t\|.$$

If $e(S,T)$ is small, then *each* point of S is close to *some* point of T, though some points of T might be far from any point of S.

We now state the basic convergence result.

Theorem 1 [Robinson (1996), Theorem 3.7] *Suppose that the following assumptions hold:*

a. With probability one, each f_n $(1 \leq n < \infty)$ is lower semicontinuous and proper.

b. With probability one, $f_n \xrightarrow{e} f_\infty$ as $n \to \infty$.

There is then a subset Γ of Ω having measure zero, with the following properties: suppose that $\omega \notin \Gamma$, let G be an open bounded set in $\mathbf{R}^k$, define for $1 \leq n \leq \infty$

$$\hat{\mu}_n(\omega) = \inf_{x \in \operatorname{cl} G} f_n(\omega, x) \qquad \text{and} \qquad \hat{M}_n(\omega) = \{x \in \operatorname{cl} G \mid f_n(\omega, x) = \hat{\mu}_n(\omega)\},$$

and assume that $\hat{M}_\infty(\omega)$ is a CLM set for $f_\infty(\omega, \cdot)$ with respect to G. Then

1. $\lim_{n \to \infty} \hat{\mu}_n(\omega) = \hat{\mu}_\infty(\omega)$, and $\hat{\mu}_\infty(\omega)$ is finite.

2. There is a finite positive integer N_ω such that for each $n \geq N_\omega$, $\hat{M}_n(\omega)$ is a nonempty, compact CLM set for $f_n(\omega, \cdot)$ with respect to G.

3. $\lim_{n \to \infty} e(\hat{M}_n(\omega), \hat{M}_\infty(\omega)) = 0$.

Theorem 1 permits us to look at sets of local minimizers that may not be global minimizers; in this sense its setting is very general. As explained in the next proposition, the assumption in Theorem 1 of the existence of a CLM set for $f_\infty(\omega, \cdot)$ can be replaced by a stronger, inf-compactness assumption.

Proposition 1 [Robinson (1996), Proposition 3.8] *Suppose that the following assumptions hold:*

a. With probability one, each f_n $(1 \leq n < \infty)$ is lower semicontinuous and proper.

b. With probability one, $f_n \xrightarrow{e} f_\infty$ as $n \to \infty$.

c. With probability one, f_∞ is proper and its set M_∞ of minimizers is nonempty and compact.
Then for almost every ω, $M_\infty(\omega)$ is a CLM set for $f_\infty(\omega, \cdot)$ with respect to some open bounded set $G(\omega)$.

Some remarks are in order. First, in general the set G of Theorem 1 depends on the sample point ω, which may cause inconvenience since we use this set to construct $\hat{\mu}_\infty(\omega)$ and $\hat{M}_\infty(\omega)$. This inconvenience can be removed by assuming that f_∞ is a deterministic function; this holds for limit functions which are expectations or steady-state performance measures.

Second, in the case of convex functions one can take G to be $\mathbf{R}^k$ in Theorem 1, *i.e.*, the localization provided by G is not necessary. We refer the interested reader to Robinson (1996) for results in the case of convex functions.

Third, if a function f is lower semicontinuous and proper but does not satisfy assumption (c) of Proposition 1, then either f has no minimizer or one can perturb f by an arbitrarily small amount at a single point to create a function whose unique minimizer has arbitrarily large norm. Since in practical optimization one always deals with inexact data, such a function would be extremely unstable from a practical point of view.

Finally, since numerical methods used in practice find solutions that are approximate, the behavior of the method when ϵ-minimizers are computed is quite important from a practical point of view. Results in Section 4 of Robinson (1996), especially Theorem 4.2, show that the behavior of the method remains unchanged in that case.

In this section we have briefly discussed the ideas behind the sample-path optimization method, presented the main convergence theorem, and commented on some issues regarding assumptions and implementation. In the next section we focus on the solution of stochastic variational inequalities and show how the sample-path idea can be extended to solve this class of problems as well.

1.3 STOCHASTIC VARIATIONAL INEQUALITIES

In this section we present an extension of sample-path optimization to solve the variational inequality (1.1) defined by a polyhedral convex set C and a function f_∞ that is an almost-sure limit of a computable sequence of random functions f_n. This situation arises in equilibrium problems in which the function f_∞ is an expectation. The function f_∞ could also represent a steady-state performance measure, or the first order necessary optimality conditions for an optimization problem. Again, the aim is to solve the variational inequality approximately by observing the functions f_n (*e.g.*, using simulation) and making the necessary computations with f_n in place of f_∞. For a deeper exposition and technical details, we refer the reader to Gürkan *et al.* (1997).

Here, the setup is one of a vector-valued stochastic process $f_n(\omega, x)$ and a vector-valued function $f_\infty(x)$, where the parameter x takes values in $\mathbf{R}^k$. Again, for all $n \geq 1$ and all $x \in \mathbf{R}^k$, the random variables $f_n(\omega, x)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$ and the k component functions of $f_n(\omega, x)$ take real values. In the simulation analogue, the f_n are estimates of f_∞ observed by a simulation run of length n. As in the sample-path optimization method, for fixed ω and n, $f_n(\omega, x)$ is a deterministic function of x, which we can compute using the method of common random numbers to fix the random number streams that are represented by ω. As in the original version, the method is simply to fix a large n (a long simulation run, to get a good estimate of f_∞) and ω, solve the deterministic variational inequality defined by C and $f_n(\omega, \cdot)$, and take the solution $x_n(\omega)$ as an approximate solution of the original problem. We will present conditions that ensure the existence of approximate solutions $x_n(\omega)$ and their closeness to the true solution x_0.

As an extension of sample-path optimization, the technique inherits many of the features of that method. Of these, two are especially advantageous. First, we can again use deterministic methods to solve the variational inequality defined by C and $f_n(\omega, \cdot)$. This is very important since (unlike the situation in stochastic optimization) there are currently no methods to solve stochastic variational inequalities unless they result from optimization problems. Second, the method is still modular; that is, it separates the simulation from the deterministic solution algorithm, so that one can use existing simulation codes and solution methods by simply writing a program that facilitates communication between the two. Modifications in system configurations or parameters can be handled easily by updating the simulation data. .

Variational inequalities can model various equilibrium phenomena of economics, operations research, and physics. Two immediate special cases are (i) a system of k nonlinear equations in k unknowns and (ii) the first-order necessary optimality conditions for a nonlinear-programming problem with continuously differentiable objective and constraint functions.

Note that not all variational inequality problems arise from optimization. In some economic equilibrium models, the lack of certain symmetry properties results in a model that is said to be *non-integrable*. In such models it is not possible to find the equilibrium prices and quantities by substituting an associated optimization problem for the variational inequality. Since the theory that we develop in Gürkan *et al.* (1997) does not require any symmetry properties, it applies to non-integrable models as well. In fact, the application to the PIES energy model that we present in Section 1.4.4, as well as the application to the European gas market of which we give an overview in Section 1.4.5, required the solution of non-integrable stochastic economic equilibrium models involving expectations.

In the particular case of an unconstrained optimization problem the method takes a special form. If we write the first-order optimality conditions for the problem, we obtain k nonlinear equations in k unknowns. Then solving the associated variational inequality would amount to finding a zero of the gradient of the objective function. In the stochastic context the approximate solution of this problem will be an estimate of a critical point of this objective function. Such a point may not be an optimizer unless certain second-order conditions are satisfied. However, when the objective function satisfies local convexity (for minimization) or concavity (for maximization) conditions around the critical point, the solution point will be an optimizer. This was the case for the option pricing problem that we briefly describe in Section 1.4.3.

To state precisely the convergence properties of the sample-path method for stochastic variational inequalities, we need to introduce a few technical concepts. To guarantee the closeness of the estimate $x_n(\omega)$ to the true solution x_0 we require a certain functional convergence property for the sequence $\{f_n\}$, namely *continuous convergence.* This property is equivalent to uniform convergence to a continuous limit on compact sets, *e.g.,* Kall (1986), and is defined as follows:

Definition 3 *A sequence f_n of extended-real-valued functions defined on $\mathbf{R}^k$* converges continuously *to an extended-real-valued function f_∞ defined on $\mathbf{R}^k$ (written $f_n \xrightarrow{C} f_\infty$) if for any $x \in \mathbf{R}^k$ and any sequence $\{x_n\}$ converging to x, one has $f_n(x_n) \to f_\infty(x)$. A sequence of functions from $\mathbf{R}^k$ into $\mathbf{R}^m$ converges continuously if each of the m component functions does so.*

To understand the rationale for requiring continuous convergence, consider sequences of functions f_n and of points x_n such that x_n solves the variational inequality defined by f_n and C and $x_n \to x$ as $n \to \infty$. Now if $f_n \xrightarrow{C} f_\infty$ then the limit point x is a solution of the limit variational inequality defined by f_∞ and C. Therefore we might reasonably use solutions of the former as estimates of the limit problem. However, although this result is useful, it unfortunately guarantees neither the existence of the solutions x_n nor their convergence. To guarantee such existence and convergence we need to impose a certain generalized nonsingularity condition. This condition is available in several equivalent forms; some of the equivalences are discussed by Dontchev and Rockafellar (1996). To make the discussion here as clear as possible, we will explain the condition in terms of a property called *strong regularity,* originally introduced by Robinson (1980). For purposes of analysis it is sometimes preferable to use another equivalent form involving *coherent orientation,* as is done for example in Gürkan *et al.* (1997).

If we define the *normal cone* of the polyhedral convex set C at a point $x \in \mathbf{R}^k$ by

$$N_C(x) = \begin{cases} \{x^* \in \mathbf{R}^k \mid \text{ for each } c \in C, \langle x^*, c - x \rangle \leq 0\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

then we can write the variational inequality (1.1) in the equivalent *generalized equation* form

$$0 \in f_\infty(x_0) + N_C(x_0). \tag{1.2}$$

Now suppose that f_∞ is Fréchet differentiable at x_0. We say that the generalized equation (1.2) (or equivalently the variational inequality (1.1)) is *strongly regular* at x_0 if there are neighborhoods X of x_0 and Y of the origin in $\mathbf{R}^k$ such that the generalized equation

$$y \in f_\infty(x_0) + df_\infty(x_0)(x - x_0) + N_C(x) \tag{1.3}$$

defines a single-valued, Lipschitzian map $x(y)$ from Y to X such that for each $y \in Y$, $x(y)$ is the unique solution in X of (1.3).

As a simple illustration of this property, we can consider the special case in which $C = \mathbf{R}^k$ (the case of nonlinear equations). It is easy to see that then (1.1) is strongly regular at x_0 if and only if $df_\infty(x_0)$ is nonsingular. In general, the strong regularity condition and its other equivalent forms provide a way to generalize the idea of nonsingularity to the case of a nontrivial set C.

The following theorem summarizes the basic convergence result; it is a simplified version of the main convergence theorem of Gürkan *et al.* (1997).

Theorem 2 *Let Φ be an open subset of $\mathbf{R}^k$ and let C be a polyhedral convex set in $\mathbf{R}^k$. Let x_0 be a point of Φ, and suppose f_∞ is a function from Φ to $\mathbf{R}^k$. Let $\{f_n \mid n = 1, 2, \ldots\}$ be random functions from Φ to $\mathbf{R}^k$ such that for all $x \in \Phi$ and all finite n the random variables $f_n(x)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume the following:*

a. With probability one, each f_n for $n = 1, 2, \ldots$ is continuous and $f_n \xrightarrow{c} f_\infty$.

b. x_0 solves the variational inequality (1.1) defined by f_∞ and C.

c. f_∞ has a strong Fréchet derivative $df_\infty(x_0)$ at x_0, and (1.1) is strongly regular at x_0.

Then there exist a positive number λ, a compact subset $C_0 \subset C \cap \Phi$ containing x_0, a neighborhood $V \subset \Phi$ of x_0, and a set $\Delta \subset \Omega$ of measure zero, with the following properties: for $n = 1, 2, \ldots$ and $\omega \in \Omega$ let

$$\xi_n(\omega) := \sup_{x \in C_0} \|f_n(\omega, x) - f_\infty(x)\|,$$

and

$$X_n(\omega) := \{x \in C \cap V \mid \textit{for each } c \in C,\ \langle f_n(\omega, x), c - x\rangle \geq 0\}.$$

For each $\omega \notin \Delta$ there is then a finite integer N_ω such that for each $n \geq N_\omega$, the set $X_n(\omega)$ is a nonempty, compact subset of the ball

$$B(x_0, \lambda\xi_n(\omega)) = \{x \mid \|x - x_0\| \leq \lambda\xi_n(\omega)\}.$$

Theorem 2 states that we can find approximate solutions of stochastic variational inequalities by the sample-path method, provided that a generalized nonsingularity condition holds. Moreover, as $n \to \infty$ the distance of the approximate solutions from an exact solution is bounded above by a constant multiple of the uniform norm of $f_n - f_\infty$ on a compact set containing the true solution.

It is possible to consider a random limit function f_∞ instead of a deterministic one in Theorem 2. However, the statement of the theorem then becomes much more complicated. Moreover, in most problems in which we envision using the method, f will be deterministic: *e.g.,* it might be the gradient of an expectation. Therefore we have kept the deterministic formulation here.

The more general version of Theorem 2, Theorem 5 of Gürkan *et al.* (1997), allows us to work not only with the the f_n's but also with small perturbations of the f_n. This is important when using a numerical method that has finite precision to solve the variational inequality defined by f_n and C. For a more detailed exposition including technical details, we refer the reader to Gürkan *et al.* (1997).

In this section we have shown how to use the sample-path methodology to solve stochastic variational inequalities, and we have sketched the theoretical justification for doing so. In the next section, we will review a number of applications, some in the optimization framework and some in that of variational inequalities, that illustrate the applicability of this technique.

1.4 APPLICATIONS

1.4.1 Stochastic PERT Networks

PERT (Program Evaluation and Review Technique) networks are used to estimate the expected duration for a project, defined as a set of activities which consume time and resources and are subject to temporal precedence relationships. In practice, an activity can often be finished in shorter (longer) periods by increasing (decreasing) the resources available to it; thus a reduction in the duration of an activity would typically come at an additional resource cost. Plambeck *et al.* (1996) considered PERT networks with up to 70 nodes and

110 stochastic arcs; each stochastic arc had a duration specified by a random number from a given probability distribution. The problem was to find the optimal parameter setting for a network in which activity duration parameters, z_i, may be changed at some cost and where these parameters were subject to linear constraints. The objective function to be minimized was the sum of the expected project completion time and a cost function expressed as a sum of terms of the form $k_i z_i^{-1}$, where k_i was the cost associated with changing z_i; this choice of objective function captures the tradeoff of increased cost against decreased project length.

The resulting objective functions were convex but possibly nonsmooth. Their subgradients could be calculated exactly by convex analysis techniques. To evaluate the objective function and its subgradient, Plambeck *et al.* simulated a set of activity lengths using a random number generator, then solved the resulting longest path network problem using the Bellman-Ford algorithm, with an obvious modification to find the longest path instead of the shortest. This calculation provided the completion time and a subgradient; they repeated this experiment a large number of times to obtain estimates of the mean subgradient and completion time values. These were then combined with the corresponding cost components to provide objective function values and subgradients for the optimization algorithm. Because of the possible nonsmoothness in the resulting sample functions and in the limit function they used the bundle-trust method of Zowe (1989) and Schramm and Zowe (1990) to solve the associated deterministic optimization problem. See Plambeck *et al.* (1996) for further details and numerical results.

1.4.2 Tandem Production Lines with Unreliable Machines

Tandem production lines consist of a number of machines in series. The material processed may be discrete entities (*e.g.* assemblies in an automobile factory) in discrete tandem (DT) lines, or it may be fluid-like in continuous tandem (CT) lines (*e.g.* for chemical production). Products start at the first machine, pass through each machine in sequence, and finally leave the system after being processed by the last machine. The time it takes a machine to process one unit of product is called the cycle time.

In many practical situations a machine may fail while it is processing, and once failed, it may take some time to be repaired; these occurrences can be modeled by specified random variables. As a consequence, various events interact and result in rather complex system dynamics. For example, if a buffer becomes full (due to failure or slow speed of a downstream machine), then upstream machines may not be able to process at their normal rate. Similarly, if a buffer is empty, then downstream machines must stop processing or slow

down. Possible decision variables include buffer capacities, cycle times, failure and repair rates of machines. The most important performance measure of a tandem line is its throughput, the amount of production completed by the last machine in unit time. Since the throughput is usually random, typically one is concerned with steady-state throughput and that was the performance measure considered in Plambeck *et al.* (1993, 1996) and Gürkan (1996).

More precisely, Plambeck *et al.* (1993, 1996) considered minimizing the reciprocal of steady-state throughput with respect to cycle times of the machines, where the cycle times were subject to constraints. On the other hand, Gürkan (1996) focused on finding the optimal buffer capacities and considered minimizing a combination of the reciprocal of steady-state throughput and a cost function; this functional form models a problem in which one wants to maximize the throughput but in which there are costs associated with increasing the buffer capacities. In both studies CT line simulations developed by Fu (1996) were used to obtain function evaluations.

Utilizing a GSMP (generalized semi-Markov process) representation provided in Suri and Fu (1994), it is possible to develop recursive expressions to measure the sensitivity of the sample throughput to cycle times or buffer capacities. Using the resulting IPA algorithms, one can compute the exact directional derivative values in a single simulation run, together with the function values. See Plambeck *et al.* (1993, 1996) and Gürkan (1996) for various details and numerical experiments on systems with up to 50 machines.

1.4.3 Option Pricing

An American call option on a dividend-paying stock is a derivative security giving the holder the right to exercise the option (by buying an underlying asset) at a prespecified exercise price at any time up to the expiration date T. Under certain assumptions about markets, including the absence of transaction costs, it is never optimal to exercise an American call option prematurely (*i.e.*, before T) unless the stock pays a dividend during the life of the option contract; see Hull (1993) and Stoll and Whaley (1993). In the case of a dividend, in order to receive the dividend it may be desirable to exercise the option just prior to an ex-dividend date, provided that the stock price then exceeds a threshold price. The choice of the threshold price is under the control of the option holder; thus one can value the option by finding its highest expected return over all possible choices of threshold prices. The problem of pricing the option then becomes that of choosing the threshold value to maximize the expected return of the option.

Fu and Hu (1995) developed an unbiased estimator for the gradient of the expected option value. Since the original problem is an unconstrained maxi-

mization problem with enough regularity, it can be solved by finding the zero of the gradient of the expected option value. In Gürkan *et al.* (1996), we provide an illustration of how these ideas together with the theory developed in Gürkan *et al.* (1997) could be used in pricing this option. Basically, we drew n samples from the underlying random stock price process and used each sample to compute one gradient value using the formulas developed in Fu and Hu (1995). By averaging these n values, we computed a sample gradient that converges to the limit gradient of which we wish to find a zero. We then used the nonsmooth Newton method of Qi and Sun (1993) to find a zero of the sample gradient. By taking a large n, we were able to price the option correctly within a penny.

The application reported in Gürkan *et al.* (1996) should be considered a simple illustration, since this problem has an analytical solution and one does not need to resort to simulation to solve it. However, it provided preliminary evidence for the implementability of the sample-path method for solving a special class of stochastic variational inequalities having practical importance.

1.4.4 Randomized PIES Model

This section describes an application of the sample-path methodology to a stochastic version of the Project Independence Evaluation System (PIES) energy model. We explain this application in more detail than the previous ones, as it provides a simple and clear illustration of how to use this method to solve stochastic variational inequalities.

The deterministic form of the PIES energy model was introduced more than 20 years ago; see Hogan (1975). This is an energy equilibrium model consisting of two sectors. The production and transportation sector uses a linear technology to produce various forms of energy. The consumption sector demands different amounts of each energy form depending on their prices.

Let q be a vector whose components are amounts of the different energy forms, and let p be the corresponding vector of prices. The production sector is described by the following linear program:

$$\begin{array}{ll} \text{minimize} & \langle c, x \rangle \\ \text{subject to} & Ax = q, \quad Bx = b, \quad x \geq 0, \end{array}$$

where x is the vector of activity levels; these are assumed to be non-negative and are associated with a cost vector c. The first set of constraints $Ax = q$ describes the linear technology used to produce the vector q of the energy forms and the next set of constraints $Bx = b$ represents the material balance constraints and bounds on the activity levels. The problem is to find a pair (p, q) such that the following properties hold simultaneously:

1. q is the vector of energy forms demanded by consumers when the prices p are in effect.

2. p is the dual (price) variable associated with the constraint that the amounts q of energy forms must be produced in the linear program solved by the production sector.

In addition, a certain relation between q and p must exist; in general this relation would be estimated by econometric methods. Following Hogan(1975), we chose this relation to be given by:

$$\ln(q_i(p)/q0_i) = \sum_{j=1}^{k} e_{ij} \ln(p_j/p0_j), \quad i = 1, \ldots, k \tag{1.4}$$

where e_{ij} are the price elasticities and where $q0$ and $p0$ are the vectors of base demand and base price respectively. If $x \in \mathbf{R}^m$, $q \in \mathbf{R}^k$, $p \in \mathbf{R}^k$, $B : \mathbf{R}^m \to \mathbf{R}^l$, and $A : \mathbf{R}^m \to \mathbf{R}^k$, then the solution is characterized by the following variational inequality: find (x, p, u) such that

$$\langle (x', p', u') - (x, p, u), f(x, p, u) \rangle \geq 0 \text{ for all } x' \in \mathbf{R}^m_+,\ p' \in \mathbf{R}^k,\ u' \in \mathbf{R}^l,$$

where u is the dual variable associated with the material balance constraints $Bx = b$ and with $q(p)$ defined by (1.4), f is given by:

$$f(x, p, u) = \begin{pmatrix} pA + uB + c \\ -Ax + q(p) \\ -Bx + b \end{pmatrix}. \tag{1.5}$$

To this well known formulation we add a twist by imagining the elasticities to be independent random variables. We assume that they are uniformly distributed with parameters given in Table 1.1; C, L, and H in the table stand for coal, light oil, and heavy oil respectively. We emphasize that the method would also work for any other choice of the distribution.

Table 1.1 Bounds on the elasticities (`eslb,esub`)

	C	L	H
C	(-0.8,-0.7)	(0.0,0.2)	(0.1,0.3)
L	(0.0,0.2)	(-0.55,-0.45)	(0.1,0.3)
H	(0.1,0.3)	(0.0,0.2)	(-0.55, -0.45)

As a consequence, $q(p)$ in (1.5) becomes a random quantity and we want to solve the limiting problem with $E[q(p)]$ instead of $q(p)$. Using the sample-path method we can do this approximately by simulating and observing a large number n of instances of e_{ij}, and solving the resulting problem with $q_n(p)$, the average of the observed values of $q(p)$.

The problem in the original form posed in Hogan (1975) can be found in the GAMS modeling language format in MCPLIB; see Dirkse and Ferris (1995b). We simply added to this program a short routine to generate the random elasticities and to average the observed $q(p)$ for each set of elasticities. The description of the various parameters used above as well as the actual values of the parameters can be found in `pies.gms` of MCPLIB. The following parameters are added: (i) `iter`, the count of random numbers (*i.e.*, the simulation length), (ii) `elast`, the elasticity, (iii) `eslb` and `esub`, the lower and the upper bounds respectively of the uniform distribution. See Özge (1997) for the details of the necessary changes to `pies.gms`.

Table 1.2 Sample-path solution of the randomized PIES problem

	$n = 50$	$n = 500$	$n = 1000$	$n =$ "∞"
p_1	11.685	11.700	11.698	11.698
p_2	13.685	13.700	13.698	13.698
p_3	15.806	15.830	15.827	15.827
p_4	16.006	16.030	16.027	16.027
p_5	11.884	11.890	11.892	11.891
p_6	12.384	12.392	12.392	12.391

Table 1.2 shows the prices of each energy form found by the sample-path method; in this table n represents the length of the simulation run used to approximate the limit function. In the solution of the variational inequalities we used the PATH solver of Dirkse and Ferris (1995a). PATH is a stabilized Newton method based on a generalization of the line search idea; it is also implemented in GAMS. The column below the heading $n =$ "∞" gives the true solution. This was found by finding the closed form of $E[q(p)]$ and using it in (1.5) instead of $q(p)$. Recall that for each $i = 1, \ldots, k$ we have $q_i(p)/q0_i =$

$\Pi_{j=1}^{k}(p_j/p0_j)^{e_{ij}}$. Then

$$\begin{aligned} E[q_i(p)] &= q0_i\ E[\Pi_{j=1}^{k}(p_j/p0_j)^{e_{ij}}] \\ &= q0_i\ \Pi_{j=1}^{k} E[(p_j/p0_j)^{e_{ij}}] \\ &= q0_i\ \Pi_{j=1}^{k} \int_{eslb(i,j)}^{esub(i,j)} (p_j/p0_j)^{e_{ij}} (esub(i,j) - eslb(i,j))^{-1} de_{ij} \\ &= q0_i\ \Pi_{j=1}^{k} (esub(i,j) - eslb(i,j))^{-1} (\ln(p_j/p0_j))^{-1} [(p_j/p0_j)^{esub(i,j)} \\ &\quad -(p_j/p0_j)^{eslb(i,j)}], \end{aligned}$$

where the second equality follows from the independence of the random variables e_{ij} for different i and j.

Solving the limiting variational inequality can be done only in special cases where one can evaluate both the limit function f and its gradient. The PIES example is one of these cases. In more general situations where we expect the sample-path method to be used, this is not possible.

1.4.5 European Natural Gas Market

Haurie *et al.* (1987) constructed a Nash-Cournot equilibrium model for analyzing certain characteristics of long term contracts in the European gas market where the main problem was to find the market price and quantity of natural gas to be produced and shipped to specified markets from different producers of natural gas over a period of time. In Gürkan *et al.* (1997), we adopted their model and partially their notation, with modifications to some parameters and functional relations in order to have a more realistic representation of this market.

The model has m players (producers and exporters of natural gas), each controlling a set of production units, n markets, and T time periods. Every player tries to maximize the net present value of its profit over the time horizon, subject to certain reserve depletion and physical capacity constraints. Let $P_j^t(Q)$ be the price of natural gas in market j during period t when the amount available in that market is Q. In contrast to the linear demand law used in Haurie *et al.* (1987), we used the following:

$$P_j^t(Q) = p0_j^t (Q/q0_j^t)^{1/e_j^t}, \tag{1.6}$$

where $p0_j^t$ and $q0_j^t$ are given parameters (the base price and base demand respectively) and e_j^t is the price elasticity of demand for natural gas in market j during period t. Since the price of natural gas given by (1.6) depends on the quantity of gas available in the market, the maximization problems faced by

every producer cannot be solved independently, and hence the overall problem should be solved as a variational inequality. This can be done by forming the first order optimality conditions of the individual maximization problems and solving them simultaneously.

Since the sample-path method is capable of handling both discrete and continuous probability distributions; we considered both cases. In the discrete case we modeled a shutdown possibility in one of the producing countries that would result in an interruption in production whereas in the continuous case the base prices and base demands of natural gas depend on the oil price and the fluctuations in the oil price follow a stochastic process. In both cases we fixed the random number streams and sampled from the underlying distributions, averaging the resulting functions to compute the sample functions f_n. The next step involved the solution of the variational inequality defined by f_n and C; for this we again used the PATH solver of Dirkse and Ferris (1995a). To observe the convergence of the solutions we considered different simulation lengths. These results and the detailed formulation of the problem can be found in Gürkan *et al.* (1997).

1.5 CONCLUSIONS

In line with the expository goal stated at the beginning of the paper, we have shown how to use sample-path methods to obtain approximate solutions of stochastic optimization problems as well as stochastic variational inequalities. We have also sketched results from successful applications to a variety of problems. Here we comment on some aspects of the methodology, and on some problems that remain to be solved.

First, one of the main advantages of this class of methods is simplicity. We use well-established techniques of simulation, together with deterministic optimization technology whose performance is by now well understood. The methods also lend themselves naturally to modular implementation, and as we have shown they can often be conveniently implemented by using modeling languages. Further, they can deal easily with constraints, and in that way they overcome one of the principal limitations of stochastic approximation.

An unanswered question about these methods is how large a sample should be chosen to get a good estimate of the limit function f_∞, and hence to get a good estimate of the solution. When f_∞ is an expectation in a static system and the sample mean construction is used to estimate it, then under mild regularity conditions one can use a certain type of central limit theorem to choose n so as to achieve a good estimate of f_∞; see Rubinstein and Shapiro (1993). In other situations, one can solve the problem for increasing values of n and observe the convergence behavior of the solutions. As employed in Plambeck *et al.* (1993,

1996) and Gürkan (1996), this approach has produced good results for large problems and complicated systems.

Good methods for solving variational inequalities almost always require evaluation of both the function f_n and its gradient; see Dirkse and Ferris (1995a) and Sellami and Robinson (1996, 1997), for example. Note that in certain cases f_n will itself be a gradient (for example, in unconstrained optimization) and so its gradient will correspond to the second derivative of another function. In more complicated problems, even if f_n is not itself a gradient it will often involve the gradients of various functions in its definition, and so second derivatives will again appear. This fact presents a potential difficulty when solving stochastic variational inequality problems for dynamic systems, since good techniques for estimating higher-order derivatives are not usually available. To overcome this difficulty seems to us an important area for future research.

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